Chapter 1 : The Foundation : Logic and proofs

Lecture 5 : Exercises & Correction

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Exercise 9

a) Use truth tables to verify the associative laws

$$(p \vee q) \vee r \equiv p \vee (q \vee r)$$

b) Use a truth table to verify the first De Morgan law

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

c) Use a truth table to verify the distributive law

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

Solution Exercise 9

DEFINITIONS

Two propositions are equivalent, if they have the same truth value for every possible combination of truth values for the variables $(p, q, r, \ldots)$.

TRUTH TABLE

A conjunction $p \wedge q$ is true, if both (sub)propositions (p and q) are true.
A disjunction $p \vee q$ is true, if either of the (sub)propositions (p or q) are true.
A negation $\neg p$ is true, if the (sub)proposition p is false.
A conditional statement $p \rightarrow q$ is true, if p is false, or if both (sub)propositions are true.
A biconditional statement $p \leftrightarrow q$ is true, if both (sub)propositions are true or if both (sub)propositions are false.

(a) The two propositions $(p \vee q) \vee r$ and $p \vee (q \vee r)$ are equivalent, because the last two columns of the following truth table are identical (except for the expression in the first row).

p	q	r	p v q	q v r	(p v q) v r	p v (q v r)
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
T	F	F	T	F	T	T
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	F	T	T	T
F	F	F	F	F	F	F

(b) The two propositions $(p \wedge q) \wedge r$ and $p \wedge (q \wedge r)$ are equivalent, because the last two columns of the following truth table are identical (except for the expression in the first row).

p	q	r	p ^ q	q ^ r	(p ^ q) ^ r	p ^ (q ^ r)
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	F	F	F
T	F	F	F	F	F	F
F	T	T	F	T	F	F
F	T	F	F	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

Exercise 10

Show that each of these conditional statements is a tautology by using truth tables.

a) $(p \wedge q) \rightarrow p$

b) $p \rightarrow (p \vee q)$

c) $\neg p \rightarrow (p \rightarrow q)$

d) $(p \wedge q) \rightarrow (p \rightarrow q)$

e) $\neg(p \rightarrow q) \rightarrow p$

f) $\neg(p \rightarrow q) \rightarrow \neg q$

Solution Exercise 10

p	q	p ^ q	(p ^ q) -> p
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

p	q	p v q	p -> (p v q)
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

p	q	not p	p -> q	not p -> (p -> q)
T	T	F	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

p	q	p -> q	not (p -> q)	not (p -> q) -> p
T	T	T	F	T
T	F	F	T	T
F	T	T	F	T
F	F	T	F	T

p	q	p ^ q	p -> q	(p ^ q) -> (p -> q)
T	T	T	T	T
T	F	F	F	T
F	T	F	T	T
F	F	F	T	T

p	q	p -> q	not (p -> q)	not q	not (p -> q) -> not q
T	T	T	F	F	T
T	F	F	T	T	T
F	T	T	F	F	T
F	F	T	F	T	T

Exercise 11

Show that these conditional statements are logically equivalent.

$\neg(p \leftrightarrow q)$ and $p \leftrightarrow \neg q$

$\neg p \leftrightarrow q$ and $p \leftrightarrow \neg q$

$(p \to r) \wedge (q \to r)$ and $(p \vee q) \to r$

Solution Exercise 11

SOLUTION

Use logical equivalence (5):

$$\neg(p \leftrightarrow q) \equiv \neg[(p \rightarrow q) \wedge (q \rightarrow p)]$$

Use De Morgan's law:

$$\equiv \neg(p \rightarrow q) \vee \neg(q \rightarrow p)$$

Use logical equivalence (2):

$$\equiv \neg(\neg p \vee q) \vee \neg(\neg q \vee p)$$

Use De Morgan's law and double negation law:

$$\equiv (p \wedge \neg q) \vee (q \wedge \neg p)$$

Use distributive law (twice):

$$\equiv ((p \wedge \neg q) \vee q) \wedge ((p \wedge \neg q) \vee \neg p)$$

$$\equiv ((p \vee q) \wedge (\neg q \vee q)) \wedge ((p \vee \neg p) \wedge (\neg q \vee \neg p))$$

Use the negation and commutative law:

$$\equiv ((p \vee q) \wedge T) \wedge (T \wedge (\neg q \vee \neg p))$$

Use identity law:

$$\equiv (p \vee q) \wedge (\neg q \vee \neg p)$$

Use commutative law (twice):

$$\equiv (q \vee p) \wedge (\neg p \vee \neg q)$$

$$\equiv (\neg p \vee \neg q) \wedge (q \vee p)$$

Use logical equivalence (2) and double negation law:

$$\equiv (p \rightarrow \neg q) \wedge (\neg q \rightarrow p)$$

Use logical equivalence (5):

$$\equiv p \leftrightarrow \neg q$$

We have thus derived that $\neg(p \leftrightarrow q)$ is logically equivalent with $p \leftrightarrow \neg q$.

SOLUTION

Use logical equivalence (5):

$$\neg(p \leftrightarrow q) \equiv \neg[(p \to q) \wedge (q \to p)]$$

Use De Morgan's law:

$$\equiv \neg(p \to q) \vee \neg(q \to p)$$

Use logical equivalence (2):

$$\equiv \neg(\neg p \vee q) \vee \neg(\neg q \vee p)$$

Use De Morgan's law and double negation law:

$$\equiv (p \wedge \neg q) \vee (q \wedge \neg p)$$

Use distributive law (twice):

$$\equiv ((p \wedge \neg q) \vee q) \wedge ((p \wedge \neg q) \vee \neg p)$$
$$\equiv ((p \vee q) \wedge (\neg q \vee q)) \wedge ((p \vee \neg p) \wedge (\neg q \vee \neg p))$$

Use the negation and commutative law:

$$\equiv ((p \vee q) \wedge T) \wedge (T \wedge (\neg q \vee \neg p))$$

Use identity law:

$$\equiv (p \vee q) \wedge (\neg q \vee \neg p)$$

Use logical equivalence (2) and double negation law:

$$\equiv (\neg p \to q) \wedge (q \to \neg p)$$

Use logical equivalence (5):

$$\equiv \neg p \leftrightarrow q$$

We have thus derived that $\neg(p \leftrightarrow q)$ is logically equivalent with $\neg p \leftrightarrow q$.

$$\neg p \leftrightarrow q \equiv p \leftrightarrow \neg q$$

Taking the right hand side:

$$p \leftrightarrow \neg q \equiv \neg(p \leftrightarrow q) \equiv \neg\left[(p \wedge q) \vee (\neg p \wedge \neg q)\right]$$
$$\equiv (p \wedge q) \wedge (\neg p \wedge \neg q) \equiv (\neg p \vee \neg q) \wedge (p - q) \equiv (p \to \neg q) \wedge (\neg p \to q)$$

Contraposative of $(p \to \neg q)$ is $(q \to \neg q)$, thus, $p \to \neg q \equiv (q \to \neg p)$.

$$(q \to \neg p) \wedge (\neg p \to q) \equiv \neg p \leftrightarrow q \quad Q.E.D$$

LOGICAL EQUIVALENCES

$$p \rightarrow q \equiv \neg p \vee q \quad (1)$$

Distributive laws:

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

De Morgans laws:

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

SOLUTION

Use logical equivalence (1):

$$(p \vee q) \rightarrow r \equiv \neg(p \vee q) \vee r$$

Use De Morgan's law:

$$\equiv (\neg p \wedge \neg q) \vee r$$

Use distributive law:

$$\equiv (\neg p \vee r) \wedge (\neg q \vee r)$$

Use logical equivalence (1):

$$\equiv (p \rightarrow r) \wedge (q \rightarrow r)$$

We have thus derived that $(p \rightarrow r) \wedge (q \rightarrow r)$ is logically equivalent with $(p \vee q) \rightarrow r$.

Exercise 12

Determine whether each of these compound propositions is satisfiable.

a) $(p \vee \neg q) \wedge (\neg p \vee q) \wedge (\neg p \vee \neg q)$
b) $(p \rightarrow q) \wedge (p \rightarrow \neg q) \wedge (\neg p \rightarrow q) \wedge (\neg p \rightarrow \neg q)$
c) $(p \leftrightarrow q) \wedge (\neg p \leftrightarrow q)$

Solution Exercise 12

a. In the truth table we notice that the compound proposition is true when p is false and q is false, thus the compound proposition is satisfiable.

p	q	p v ~q	~ p v q	~p v ~q	Compound propisition
T	T	T	T	F	F
T	F	T	F	T	F
F	T	F	T	T	F
F	F	T	T	T	T

b. In the truth table we notice that the compound proposition is always false, thus the compound proposition is unsatisfiable.

p	q	p -> q	p -> ~q	~p -> q	~p -> ~q	Compound propisition
T	T	T	F	T	T	F
T	F	F	T	T	T	F
F	T	T	T	T	F	F
F	F	T	T	F	T	F

c. In the truth table we notice that the compound proposition is always false, thus the compound proposition is unsatisfiable.

p	q	p <-> q	~p <-> q	Compound propisition
T	T	T	F	F
T	F	F	T	F
F	T	F	T	F
F	F	T	F	F

Exercise 13

Let P (x) be the statement "x can speak Russian" and let Q(x) be the statement "x knows the computer language C++." Express each of these sentences in terms of P (x), Q(x), quantifiers, and logical connectives. The domain for quantifiers consists of all students at your school.

a) There is a student at your school who can speak Russian and who knows C++.

b) There is a student at your school who can speak Russian but who doesn't know C++.

c) Every student at your school either can speak Russian or knows C++.

d) No student at your school can speak Russian or knows C++.

Solution Exercise 13

$$\neg\exists x(P(x) \lor Q(x))$$

$$\neg\exists x(P(x) \lor Q(x))$$

$$\neg\exists x(P(x) \lor Q(x))$$

$$\neg\exists x(P(x) \lor Q(x))$$

Exercise 14

Let P(x) be the statement "x=x^2" If the domain consists of the integers, what are these truth values?

a) $P(0)$ **b)** $P(1)$ **c)** $P(2)$
d) $P(-1)$ **e)** $\exists x P(x)$ **f)** $\forall x P(x)$

Solution Exercise 14

(a) $P(0)$ is true, because the corresponding statement $0 = 0^2$ is true.

(b) $P(1)$ is true, because the corresponding statement $1 = 1^2$ is true.

(c) $P(2)$ is false, because the corresponding statement $2 = 2^2$ is false (since $2^2 = 4$).

(d) $P(-1)$ is false, because the corresponding statement $-1 = (-1)^2$ is false (since $(-1)^2 = +1$).

(e) $\exists x P(x)$ is true, because we know that $P(x)$ is true for $x = 0$ (due to part (a)) and thus there exists and x for which $P(x)$ is true.

(f) $\forall x P(x)$ is false, because we know that $P(x)$ is false for $x = 2$ (due to part (c)) and thus it is not true that $P(x)$ is true for all possible values of x.

Exercise 15

a) Use a direct proof to show that the sum of two odd integers is even.

b) Use a contraposition proof to prove that if n is an integer and n2 is odd, then n is odd.

Solution Exercise 15

TO PROOF: The sum of two odd integers is even.

PROOF

Properties odd and even integer:
If x is an odd integer, then there exists an integer y such that $x = 2y + 1$.
If x is an even integer, then there exists an integer y such that $x = 2y$.

Let a and b be odd integers, then there exists integers y and z such that:

$$a = 2y + 1$$
$$b = 2z + 1$$

We are interested in the sum of the two odd integers:

$$a + b = 2y + 1 + 2z + 1 = 2y + 2z + 2 = 2(y + z + 1)$$

Since y and z are integers, $y + z + 1$ is also an integer and thus $a + b$ is even (using the above property for even integers).
□

Exercise 16

Determine the truth value of each of these statements if the domain consists of all integers.

a) $\forall n(n + 1 > n)$

b) $\exists n(2n = 3n)$

c) $\exists n(n = -n)$

d) $\forall n(3n \leq 4n)$

Solution Exercise 16

a) $\forall n(n+1 > n)$
True n+1 will always be greater than n (even negative numbers)

b) $\exists n(2n = 3n)$
True n can be 0; (0 = 0)

c) $\exists n(n = -n)$
True n can be 0; (0 = 0)

d) $\forall n(3n \leq 4n)$
False, n can be -1. ($-3 \leq -4$) is not true

x can be 1,2 or 3 and all values have to be possible.

Exercise 17

Suppose the domain of the propositional function $P(x, y)$ consists of pairs x and y, where x is 1, 2, or 3 and y is 1, 2, or 3. Write out these propositions using disjunctions and conjunctions.

a) $\exists x\, P(x, 3)$ **b)** $\forall y\, P(1, y)$

c) $\exists y \neg P(2, y)$ **d)** $\forall x\, \neg P(x, 2)$

Solution Exercise 17

a) $P(1,3) \vee P(2,3) \vee P(3,3)$

x can be 1,2 or 3 and there has to exists 1 value out of the three for the propositional function.

b) $P(1,1) \wedge P(1,2) \wedge P(1,3)$

y can be 1,2 or 3 and all values have to be possible.

c) $\neg P(2,1) \vee \neg P(2,2) \vee \neg P(2,3)$

y can be 1,2 or 3 and there has to exists 1 value out of the three for the propositional function.

d) $\neg P(1,2) \wedge \neg P(2,2) \wedge \neg P(3,2)$

x can be 1,2 or 3 and all values have to be possible.

Exercise 18

Find the argument form for the following argument and determine whether it is valid. Can we conclude that the conclusion is true if the premises are true?

If Socrates is human, then Socrates is mortal.
Socrates is human.

$\therefore$ Socrates is mortal.

Exercise 19

Find the argument form for the following argument and determine whether it is valid. Can we conclude that the conclusion is true if the premises are true?

If George does not have eight legs, then he is not a spider.
George is a spider.

$\therefore$ George has eight legs.

Solution Exercise 18

SOLUTION

Let us assume:

$p =$"Socrates is human"
$q =$"Socrates is mortal"

We can then rewrite the given argument as:

$$\begin{array}{rc} & p \to q \\ & p \\ \hline \therefore & q \end{array}$$

The order of the statements in the argument does not matter, as long as the conclusion remains at the very bottom.

$$\begin{array}{rc} & p \\ & p \to q \\ \hline \therefore & q \end{array}$$

We then note that the conclusion is true, because the argument is identical to rule of modus ponens.

INTERPRETATION SYMBOLS

Conditional statement $p \to q$: if p, then q

RULES OF INFERENCE

Modus ponens

$$\begin{array}{rc} & p \\ & p \to q \\ \hline \therefore & q \end{array}$$

Solution Exercise 19

SOLUTION

Let us assume:

$p =$ "George has eight legs"
$q =$ "George is a spide"

We can then rewrite the given argument using the above interpretations:

$$\begin{array}{rc} & \neg p \to \neg q \\ & q \\ \hline \therefore & p \end{array}$$

We can use the double negation law:

$$\begin{array}{rc} & \neg p \to \neg q \\ & \neg(\neg q) \\ \hline \therefore & \neg(\neg p) \end{array}$$

The order of the statements in the argument does not matter, as long as the conclusion remains at the very bottom.

$$\begin{array}{rc} & \neg(\neg q) \\ & \neg p \to \neg q \\ \hline \therefore & \neg(\neg p) \end{array}$$

We then note that the conclusion is true, because the argument is the rule of modus tollens.

INTERPRETATION SYMBOLS

Negation $\neg p$: not p

Conditional statement $p \to q$: if p, then q

LOGICAL EQUIVALENCES

Double negation law:

$$\neg(\neg p) \equiv p$$

RULES OF INFERENCE

Modus tollens

$$\begin{array}{rc} & \neg q \\ & p \to q \\ \hline \therefore & \neg p \end{array}$$